Fundamental Theorem of Algebra

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Abstract

We give a very short proof of the Fundamental Theorem of Algebra using a one-parameter deformation and a discriminant-locus argument.

1 Main Theorem

Theorem 1 (Fundamental Theorem of Algebra)**.** *Every polynomial of degree n with complex coefficients has exactly n complex roots, counted with multiplicity.*

For a given polynomial $p(x) = x^n + p_{n-1}x^{n-1} + \cdots + p_0$, we consider the family of polynomials $p_t = tp + (1-t)q$ for $t \in \mathbb{C}$. Where $p_0 = q$ is a polynomial with n distinct roots, e.g. $q = x^n - 1$.

Proof. Let $X \subset \mathbb{C}$ be the set of parameters t where p_t has at least one root. By construction $0 \in X$. We will show that $1 \in X$, by deforming the roots of q along p_t using Resultant techniques and the Implicit Function theorem, and leveraging the Cauchy bound on root location to show stability of roots under limits with bounded coefficients. This shows that p has a single root r, by splitting off a linear factor $x - r$ and induction on the degree of p we conclude that p has n roots.

The set $X \subset \mathbb{C}$ is closed:

Let t_k be a sequence in X converging to a point $t \in \mathbb{C}$. We need to show that $t \in X$. As t_k is bounded, we know that the coefficients of p_{t_k} are bounded by a constant M. By the Cauchy bound, all roots of p_{t_k} lie within the disk of radius $1 + \max_i |p_{t_k,i}| \leq M + 1$. Let r_k be a convergent sequence of roots of p_{t_k}, and let $r = \lim_{k\to\infty} r_k$. Then $\lim_{k\to\infty} p_{t_k}(r_k) = p_t(r)$ by continuity of p_t, but $p_{t_k}(r_k) = 0$, so $p_t(r) = 0$.

Now consider $\Delta(t) = Res(p_t, p_t')$ the discriminant of the polynomial p_t.

Recall that the Resultant $Res(p,q)$ is a polynomial in the coefficients of the polynomials p and q and that $Res(p,q) = 0$ if and only if the polynomials p and q have a common factor ($gcd(p,q) \neq 1$). In particular $\Delta(t)$ is a polynomial in t, with $\Delta(0) \neq 0$ as q has n distinct roots.

Let $D = \{t \in \mathbb{C} \mid \Delta(t) = 0\}$, and let $U = \mathbb{C} \setminus D$. As complement of a finite set, U is connected open and dense in $\mathbb{C}$.

The set $X \cap U \subset \mathbb{C}$ is open:

Let $t \in X \cap U$, we need to show that there is a neighborhood N of t such that $N \subset X$.

Since $t \in X$ we know that p_t has at least one root r. As $t \in U$ we know that $p_t'(r) \neq 0$, since otherwise p_t and p_t' would have a common factor. By the implicit function theorem applied to $F(x,t) = p_t(x)$, there exists a neighborhood N of t and a function $r(\tau)$ defined on N, with $r(t) = r$ and $p_\tau(r(\tau)) = 0$ for all $\tau \in N$. This shows that $N \subset X$.

We conclude that $X \cap U$ is open and closed in U. As U is connected, it follows that $X \cap U = U$. Thus, X contains the open dense set U, which implies that X is all of $\mathbb{C}$, and in particular $1 \in X$. ☐

2 History and Related Work

The *Fundamental Theorem of Algebra* has a long and contested history, with early attempts by d'Alembert, Euler, and Laplace, and more substantial proofs by Gauss (1799, 1816, 1849) and Argand (1806/1813). For surveys and historical accounts, see Remmert's exposition [Rem91], the monograph of Fine–Rosenberger [FR97], and Gilain's historical study [Gil91].

Since then, the theorem has continued to attract new proofs, ranging from analytic (Liouville, Rouché) to topological (winding numbers, degree arguments), constructive (Kneser [Kne39], Richman [Ric00]), and elementary (Rio Branco de Oliveira [dO12], Basu [Bas21]). This ongoing stream of contributions underscores both the theorem's centrality and its pedagogical appeal.

The proof given here exploits a discriminant complement arguments: one shows that, away from parameters where the discriminant vanishes, roots continue locally by the implicit function theorem, while global existence follows from connectedness and a compactness/closedness input (here provided by Cauchy's root bound). The same strategy has been independently used earlier by Pukhlikov–Pushkar (see [Con]) and in a recent blog post by Litt [Lit16]. The point of view taken here is a particularly compact, one-dimensional variant, restricting the argument to a single pencil $p_t = (1 - t)q + tp$.

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